Lecture 13: Solving integrals by substitution

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$$\int \frac{1}{x} dx = \ln|x| + C$$

(don't forget absolute value)

What about:

$$\int \frac{1}{x+1} dx = \ln|x+1| + C?$$

Test: Check the guess by taking its derivative.

$$(\ln|x+1| + C)' = (\ln|x+1|)' + (C)'$$
$$= \frac{1}{x+1} + 0 = \frac{1}{x+1}$$

✓ works

Why?

$\frac{1}{x+1}$ is a composition of functions (recall chapter 1):

$f(x) = \frac{1}{x}, \qquad g(x) = x + 1$

then, $f\big(g(x)\big) = \frac{1}{x+1}$

write $\int \frac{1}{x+1} dx$ as $\int f\big(g(x)\big)\, dx$

which $= \int f\big(g(x)\big) * g'(x) \; dx$

If $(s \circ t)(x)$, then by the chain rule:

$\big((s \circ t)(x)\big)' = s'\big(t(x)\big) * t'(x)$

eg. $(\sin(x^2))' = \cos(x^2) * 2x$, where $s(x) = \sin(x)$
$t(x) = x^2$

Guess: the antiderivative of a composition of functions is also a composition of functions. Determine s and t to make it work.

By the **Fundamental Theorem of Calculus**, the antiderivative of $(f \circ g)(x) * g'(x)$ has to have the property that its derivative is again $(f \circ g)(x) * g'(x)$.

Compare $f\big(g(x)\big) * g'(x)$ with $s'\big(t(x)\big) * t'(x)$

Set $t(x) = g(x)$, $f(x) = s'(x)$

the guess for the antiderivative of f(g(x))*g'(x) was s(t(x)). So, we need to $\underbrace{\text{find s(x) from s'(x).}}_{\text{integration}}$

in example:

$$\int \frac{1}{x+1} dx = \int f\big(g(x)\big) * g'(x)\, dx \quad \text{with} \quad f(x) = \frac{1}{x}, \qquad g(x) = x + 1$$

Now we know, an antiderivative of $\int f\big(g(x)\big) * g'(x)$ is given by:

$(F \circ g)(x) + C$, where $F(x)$ is an antiderivative of $f(x)$

$f(x) = \frac{1}{x}$

$F(x) = \ln|x|$

$(F \circ g)(x) = \ln|x+1| + C$

This also works if $g'(x) \neq 1$!

Example:

$\int 2x\sqrt{1+x^2}\,dx$ find f, g such that $f(g(x)) * g'(x) = 2x\sqrt{1+x^2}$

if $g(x) = 1 + x^2$
$g'(x) = 2x$
$f(x) = \sqrt{x}$

$= \int f(g(x)) * g'(x)\,dx$
$= F(g(x)) + C$ where F is an antiderivative of f
$= \frac{2}{3}(1+x^2)^{\frac{1}{2}} + C$

The Substitution Rule

$\int f(g(x)) * g'(x)\,dx = F(g(x)) + C$

or

if $u = g(x)$, then if f is continuous:

$\int f(g(x)) * g'(x)\,dx = \int f(u)\,du$

by the chain rule: $(F(g(x)+C)' = F'(g(x)) * g'(x)$
$= f(g(x)) * g'(x)$

Examples

$\int 5e^{5x}\,dx$

1 set $u = 5x$

$\frac{du}{dx} = 5$

2 $\frac{du}{5} = dx$

3

4 $\int \cancel{5}e^u * \frac{du}{\cancel{5}}$

$\int e^u\,du$

5 $= e^{5x} + C$

$\int \sqrt{2x+1}\,dx = \left| \begin{array}{l} u = 2x+1 \\ \frac{du}{dx} = 2,\ dx = \frac{du}{2} \end{array} \right| = \int \sqrt{u} * \frac{1}{2}\,du$

$= \frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{2} * \frac{(u^{\frac{1}{2}+1})}{\frac{1}{2}+1} + C$

$= \frac{1}{2} * \frac{2}{3}u^{\frac{3}{2}} + C$

$= \frac{1}{3}(2x+1)^{\frac{3}{2}} + C$

We can use this technique for definite integrals as well.

$\int_0^1 2x\sin(x^2+1)\,dx$ $f(x) = \sin(x)$

$g(x) = x^2 + 1$
$g'(x) = 2x$

$= \left| \begin{array}{c} u = x^2+1 \\ \frac{du}{dx} = 2x, dx = \frac{du}{2x} \end{array} \right| = \int_0^1 \cancel{2x}\sin(u) * \frac{\text{du}}{\cancel{2\text{x}}} = \int_0^1 \sin(u)\,du$

be careful
these are bounds
for x

$= -\cos(1^2+1) - (-\cos(0^2+1))$

$= -\cos(2) + \cos(1)$

Sidenote: if you want to separately calculate the boundaries with respect to u, rather than x:

1) we've set $u = x^2 + 1$
2) our bounds, a and b, are 0 and 1
3) so, $a = (o)^2 + 1 = 1$, and $b = (1)^2 + 1 = 2$